THE GENUS OF SUBGRAPHS OF K_8

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ABSTRACT

A graph of genus 2 which is irreducible with respect to this property must have at least eight vertices. It is shown here that there are exactly three such 2-irreducible graphs having eight vertices. The description of these three graphs, together with the Kuratowski theorem, leeds to a determination of the genus of each graph having fewer than nine vertices.

Let M_k denote the closed orientable 2-manifold of genus k. The *genus* of a graph G, denoted by $\gamma(G)$, is the smallest k for which G can be embedded in M_k. A graph G is *n-irreducible* if and only if $\gamma(G) = n$ and $\gamma(H) < n$ for all proper subgraphs H of G, or equivalently if $\gamma(G) = n$ and $\gamma(G - e) = n - 1$ for each edge e of G. It is easy to see that a graph G has genus greater than or equal to n if and only if G contains an n-irreducible subgraph. Thus the Kuratowski characterization of planar graphs [2] may be viewed as a proof of the fact that each 1-irreducible graph is homeomorphic to the complete graph on five vertices or to the complete 3×3 bipartite graph.

Considerable effort has been spent in attempts to characterize graphs of a given positive genus. In particular, much attention has been paid to the class of 2-irreducible graphs in an effort to characterize graphs of genus 1. It is now known [1] that there are at least 320 nonhomeomorphic members of this class. Because of the continued interest in this subject, it seems worthwhile to record some results obtained several years ago which identify the 2-irreducible graphs having the smallest possible number of vertices and lead to a determination of the genus of each graph having fewer than nine vertices.

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In order to describe these smallest 2-irreducible graphs as well as certain auxiliary graphs, we adopt the following notation. The complete graph on n vertices is denoted by K_n and the complete $m \times n$ bipartite graph by $K_{m,n}$. Also L_n denotes a simple path of length n and C_n a simple circuit of length n. By $K_n - G$ we mean the graph obtained by deleting from K_n the edges of a subgraph isomorphic to G. For graphs G and H, $G \cdot H$ denotes a graph obtained by identifying one vertex of G with one vertex of H, and $\{G, H\}$ denotes the (disconnected) graph whose components are G and H.

It is well known that $\gamma(K_7) = 1$, so each graph with at most 7 vertices has genus less than or equal to one. But $\gamma(K_8) = 2$ and therefore K_8 contains at least one 2-irreducible subgraph. The following results show that there are exactly three 2-irreducible subgraphs of K_8, namely: $B_1 = K_8 - K_3$, $B_2 = K_8 - \{K_2, K_2, L_2\}$, and $B_3 = K_8 - K_{2,3}$.

To show this we also consider the following eight graphs: $T_1 = K_8 - \{K_2, K_2, K_2, K_2\}$, $T_2 = K_8 - K_2 \cdot K_3$, $T_3 = K_8 - \{K_2, K_{1,3}\}$, $T_4 = K_8 - \{K_2, L_3\}$, $T_5 = K_8 - \{L_2, L_2\}$, $T_6 = K_8 - \{K_2, K_3\}$, $T_7 = K_8 - K_{1,4}$, and $T_8 = K_8 - C_5$. The T_i for $1 \leqq i \leqq 7$ are the graphs of triangulations of the torus. The remaining graph, T_8, also has genus 1. These facts together with some simple constructions yield the following result.

THEOREM 1. *Each subgraph of K_8 which does not contain any of the graphs B_1, B_2 or B_3 as a (proper or improper) subgraph is itself a subgraph of T_j for some $j \in \{1, 2, \cdots, 8\}$, and hence has genus less than or equal to one. In particular. for any edge e of B_i, $\gamma(B_i - e) \leqq 1$ for $i = 1$, 2,* or 3.

All that remains to be shown then is that $\gamma(B_i) = 2$ for $i = 1, 2, 3$. For this, let the vertex set of K_8 be $V = \{v_1, v_2, \cdots, v_8\}$. For $X \subseteq V$, let $\langle X \rangle$ be the subgraph of K_8 whose vertex set is X and whose edges are all those edges of K_8 joining two vertices in X.

THEOREM 2. $\gamma(B_i) = 2$ *for* $i = 1, 2, 3$.

PROOF. That $\gamma(B_1) \geqq 2$ follows easily from Euler's formula. But $B_1 \subseteq K_8$ implies that $\gamma(B_1) \leqq 2$.

We consider B_2 and B_3 separately.

Let B_2 be obtained by deleting the edges (v_1, v_2), (v_2, v_3), (v_4, v_5), and (v_7, v_8)

from K_8. Since $K_5 \subseteq B_2 \subseteq K_8$, we have $1 \leqq \gamma(B_2) \leqq 2$. Suppose $\gamma(B_2) = 1$. Then there is an embedding ξ of B_2 in M_1. By the Euler formula, ξ must provide a triangulation of M_1. Consider the triangles which are boundaries of 2-cells, the components of $M_1 - \xi(B_2)$, and have $\xi(v_2)$ as a vertex. There are five of these, two of which also have $\xi(v_6)$ as a vertex since the degree of v_6 in B_2 is seven. The remaining three bounding triangles having $\xi(v_2)$ as a vertex must be among the images of $\langle\{v_4, v_2, v_7\}\rangle$, $\langle\{v_4, v_2, v_8\}\rangle$, $\langle\{v_5, v_2, v_7\}\rangle$, and $\langle\{v_5, v_2, v_8\}\rangle$. The image of one of these four does not bound a component of $M_1 - \xi(B_2)$. We may assume that the image of $Y = \langle\{v_4, v_2, v_8\}\rangle$ is such a nonbounding triangle. The subgraph of B_2 which is $\langle\{v_1, v_3, v_5, v_6, v_7\}\rangle$ is isomorphic to K_5 and is disjoint from Y. Now suppose $\xi(Y)$ is homotopic to a point, i.e. some component of $M_1 - \xi(T)$ is an open 2-cell P. Since $B_2 - Y$ is connected and contains a K_5, $B_2 - Y$ lies entirely outside of P. But this implies that $\xi(Y)$ bounds a component of $M_1 - \xi(B_2)$. From this contradiction one concludes that no component of $M_1 - \xi(Y)$ is an open 2-cell. But then by 3.4 of [3], there is a manifold N such that the closed set $\langle\{v_1, v_3, v_5, v_6, v_7\}\rangle$ can be embedded in N with $\gamma(N) < \gamma(M_1)$. Since $\gamma(K_5) = 1$, this is also impossible.

Let B_3 be obtained by deleting the edges (v_i, v_j), $i = 1,2,3$; $j = 4,8$, from K_8. As for B_2, $1 \leqq \gamma(B_3) \leqq 2$, and we suppose that there is an embedding ξ of B_3 in M_1. Consider the triangles in B_3 whose sets of vertices are respectively $\{v_4, v_5, v_8\}$, $\{v_4, v_6, v_8\}$, and $\{v_4, v_7, v_8\}$. Since each of these triangles contains the edge (v_4, v_8), at most two of their images are the boundaries of components of $M_1 - \xi(B_3)$. We may assume that the triangle with vertices $\xi(v_4)$, $\xi(v_5)$, and $\xi(v_8)$ is not such a boundary. Note that $B_3 - \langle\{v_4, v_5, v_8\}\rangle$ is connected and contains a copy of K_5. By an argument similar to that above, the triangle with vertices $\xi(v_4)$, $\xi(v_5)$, and $\xi(v_8)$ is not homotopic to a point. As for B_2, it now follows that $\gamma(B_3) = 2$.

COROLLARY. *For each subgraph H of K_8 we have one of the following.*

$$\gamma(H) = \begin{cases} 0 & \text{if } H \text{ does not contain a subgraph homeomorphic to either Kuratowski graph.} \\ 1 & \text{if } H \text{ contains a subgraph homeomorphic to one of the Kuratowski graphs, but does not contain any of } B_1, B_2, \text{ or } B_3. \\ 2 & \text{if } H \text{ contains any of } B_1, B_2, \text{ or } B_3. \end{cases}$$

References

1. G. Haggard, *2-Irreducible graphs* (unpublished).

2. K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. **15** (1930), 271–283.

3. J. W. T. Youngs, *Minimal imbeddings and the genus of a graph*, J. Math. Mech. **12** (1963), 303–315.

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